

# ON THE EXISTENCE OF WEAK SOLUTIONS OF A THERMISTOR SYSTEM WITH $p$ -LAPLACIAN TYPE EQUATION: THE UNSTEADY CASE

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ABSTRACT. Let  $\Omega \subset \mathbb{R}^n$  ( $n = 2$  or  $n = 3$ ) be a bounded domain. We consider the thermistor system

$$(1) \quad \nabla \cdot \mathbf{J} = 0, \quad (2) \quad \frac{\partial u}{\partial t} + \nabla \cdot \mathbf{q} = f(x, t, u, \nabla \varphi) \quad \text{in } \Omega \times ]0, T[,$$

where (1) is a  $p$ -Laplace type equation for  $\varphi$  ( $u$  = temperature,  $\varphi$  = electrostatic potential). We prove the existence of a weak solution  $(\varphi, u)$  of (1)–(2) under mixed boundary conditions for  $\varphi$ , and a Robin boundary condition and an initial condition for  $u$ .

## 1. INTRODUCTION

Let  $\Omega \subset \mathbb{R}^n$  ( $n = 2$  or  $n = 3$ ) be a bounded domain with Lipschitz boundary  $\partial\Omega$ , and set  $Q_T = \Omega \times ]0, T[$  ( $0 < T < +\infty$ ).

Let  $\mathbf{J}$  and  $\mathbf{q}$  denote the electric current field density and the heat flux, respectively, of a thermistor occupying the domain  $\Omega$  under unsteady operating conditions. Then the balance equations for the electric current and the heat flow within the thermistor material are the following two PDEs

$$\nabla \cdot \mathbf{J} = 0, \quad \frac{\partial u}{\partial t} + \nabla \cdot \mathbf{q} = f(x, t, u, \nabla \varphi) \quad \text{in } Q_T,$$

where  $\varphi = \varphi(x, t)$  and  $u = u(x, t)$  represent the electrostatic potential and the temperature, respectively (see, e.g., [25, Chap. 8]).

We make the following constitutive assumptions on  $\mathbf{J}$  and  $\mathbf{q}$

$$\mathbf{J} = \sigma(u, |\mathbf{E}|) \mathbf{E} \quad \text{Kirchhoff's law,} \quad \mathbf{q} = -\kappa(u) \nabla u \quad \text{Fourier's law,}$$

where

$$\mathbf{E} = -\nabla \varphi \quad \text{density of the electric field,}$$

$$\sigma = \sigma(u, |\mathbf{E}|) \quad \text{electrical conductivity,}$$

$$\kappa = \kappa(u) \quad \text{thermal conductivity.}$$

With these notions the above system of PDEs takes the form

$$(1) \quad -\nabla \cdot (\sigma(u, |\nabla \varphi|) \nabla \varphi) = 0 \quad \text{in } Q_T,$$

$$(2) \quad \frac{\partial u}{\partial t} - \nabla \cdot (\kappa(u) \nabla u) = f(x, t, u, \nabla \varphi) \quad \text{in } Q_T.$$

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The function  $f = f(x, t, u, \nabla\varphi)$  represents a heat source that will include the Joule heat  $\mathbf{J} \cdot \mathbf{E}$  as special case.

We supplement system (1)–(2) by boundary conditions for  $\varphi$  and  $u$ , and an initial condition for  $u$ . Without any further reference, throughout the paper we assume

$$\partial\Omega = \Gamma_D \cup \Gamma_N \text{ disjoint, } \Gamma_D \text{ non-empty, open.}$$

Define

$$\Sigma_D = \Gamma_D \times ]0, T[, \quad \Sigma_N = \Gamma_N \times ]0, T[.$$

We then consider the conditions

$$(3) \quad \varphi = \varphi_D \text{ on } \Sigma_D, \quad \mathbf{J} \cdot \mathbf{n} = 0 \text{ on } \Sigma_N,$$

$$(4) \quad \mathbf{q} \cdot \mathbf{n} = g(u - h) \text{ on } \partial\Omega \times ]0, T[,$$

$$(5) \quad u = u_0 \text{ in } \Omega \times \{0\}$$

( $\mathbf{n}$  = unit outward normal to  $\partial\Omega$ ). The first condition in (3) means that there is an applied voltage  $\varphi_D$  along  $\Sigma_D$ , whereas the second condition characterizes electrical insulation of the thermistor along  $\Sigma_N$ . The Robin boundary condition (4)<sup>1)</sup> means that the flux of heat through  $\partial\Omega \times ]0, T[$  is proportional to the temperature difference  $u - h$ , where  $g$  denotes the thermal conductivity of the surface  $\partial\Omega$  of the thermistor, and  $h$  represents the ambient temperature (cf. [8], [10], [15], [24] (nonlinear boundary conditions)).  $\square$

We consider the following prototype for electrical conductivities  $\sigma$  in (1). Let  $\sigma_0 : \mathbb{R} \rightarrow \mathbb{R}_+$ <sup>2)</sup> be a continuous function such that

$$0 < \sigma_* \leq \sigma_0(u) \leq \sigma^* \quad \forall u \in \mathbb{R} \quad (\sigma_*, \sigma^* = \text{const}).$$

Let  $\delta = \text{const} > 0$  and let  $1 < p < +\infty$ . We consider

$$(6) \quad \sigma = \sigma(u, |\xi|) = \sigma_0(u)(\delta + |\xi|^2)^{(p-2)/2}, \quad \xi \in \mathbb{R}^n.$$

Here, the factor  $\sigma_0(u)$  describes the thermal dependence of the electrical conductivity  $\sigma$  of the thermistor material. We obtain

$$\mathbf{J} = \sigma(u, |\mathbf{E}|) \mathbf{E} = -\sigma_0(u)(\delta + |\nabla\varphi|^2)^{(p-2)/2} \nabla\varphi$$

and equ. (1) is of  $p$ -Laplace type

$$-\nabla \cdot (\sigma_0(u)(\delta + |\nabla\varphi|^2)^{(p-2)/2} \nabla\varphi) = 0.$$

If  $p = 2$  and  $f = \mathbf{J} \cdot \mathbf{E}$  (Joule heat), then (1)–(2) represent the well-known thermistor system (see, e.g., [1], [9]).

To make things clearer, let  $I = |\mathbf{J}|$  and  $V = |\mathbf{E}|$  denote current and voltage, respectively, in an electrical conductor. With  $\sigma$  as in (6) we obtain the current-voltage characteristic

$$(7) \quad I = I(u, V) = \sigma_0(u)(\delta + V^2)^{(p-2)/2} V.$$

If  $1 < p \leq 2$  ( $p$  “near to 1”), then this characteristic describes approximately the current-voltage relations of transistors (cf., e.g., [16], [27, Chap. 6.2.2]). In particular, if  $p = 1$ , then (7) is widely used to model the effect of saturation of current under high electric fields in certain transistors. For details see, e.g., [21, Chap. 2.5].

<sup>1)</sup>This boundary condition is also called Newton’s cooling law.

<sup>2)</sup> $\mathbb{R}_+ = [0, +\infty[$ .

**Remark 1.** (The case  $2 \leq p < +\infty$ ). In [13] (formula (1),  $\alpha \geq 1$ ;  $p = \alpha + 1$  in our notation), the authors consider current-voltage characteristics of the form

$$(8) \quad I = I(u, V) = \sigma_0(u) V^{p-2}, \quad 2 \leq p < +\infty$$

for modeling thermistor-like self-heating effects in organic semiconductors. These characteristics can be approximated by (7) for sufficiently small  $\delta > 0$ .

For the steady case of (1)–(4) and coefficients  $\sigma_0 = \sigma_0(x, u)$  and  $2 < p < +\infty$  in (8), the existence of weak solutions for the case of two dimensions has been proved for the first time in [18]. This result was extended to the case of measurable exponents  $2 \leq p(x) < +\infty$  ( $x \in \Omega$ ) in [14]. An extension of the latter result has been recently presented in [7].  $\square$

We present a prototype for functions  $f = f(x, t, u, \xi)$  on the right hand side of (2). For  $(x, t, u, \xi) \in Q_T \times \mathbb{R} \times \mathbb{R}^n$  we consider

$$(9) \quad f(x, t, u, \xi) = \eta(x, t, u, -a(u, -\xi)\xi)\sigma(u, |\xi|)|\xi|^2,$$

where  $\sigma = \sigma(u, |\xi|)$  is as in (6) and

$$\begin{cases} \eta = \eta(x, t, u, \hat{\xi}) : Q_T \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}_+ \text{ is Carathéodory,} \\ 0 \leq \eta(x, t, u, \hat{\xi}) \leq \eta_1 = \text{const} \quad \forall (x, t, u, \hat{\xi}) \in Q_T \times \mathbb{R} \times \mathbb{R}^n, \\ a = a(u, \xi) : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R} \text{ is continuous.} \end{cases}$$

Writing  $\xi = -\mathbf{E}$ <sup>3)</sup> we obtain

$$f(x, t, u, -\mathbf{E}) = \eta(x, t, u, a(u, \mathbf{E})\mathbf{E})\mathbf{J} \cdot \mathbf{E}.$$

The condition for  $\eta$  and  $a$  can be specified in several ways, e.g.,  $a(u, \xi) = \sigma(u, |\xi|)$ . Then  $\eta$  may be considered as depending on  $\mathbf{J}$ . In particular, if  $0 < \eta(x, t, u, \mathbf{J}) < 1$ , then the source term  $f(x, t, u, -\mathbf{E})$  in (2) models a loss of Joule heat (cf. [18] for more details).  $\square$

Our paper is organized as follows

2. Weak formulation of (1)–(5). Statement of the main result
3. Proof of the main result
  - 3.1 Existence of an approximate solution
  - 3.2 A-priori estimates
  - 3.3 Passage to the limit  $\varepsilon \rightarrow 0$

References

## 2. WEAK FORMULATION OF (1)–(5). STATEMENT OF THE MAIN RESULT

We introduce the notations which will be used in what follows.

By  $W^{1,p}(\Omega)$  ( $1 \leq p < +\infty$ ) we denote the usual Sobolev space. Define

$$W_{\Gamma_D}^{1,p}(\Omega) = \{v \in W^{1,p}(\Omega); v = 0 \text{ a.e. on } \Gamma_D\}.$$

This space is a closed subspace of  $W^{1,p}(\Omega)$ . Throughout the paper, we consider  $W_{\Gamma_D}^{1,p}(\Omega)$  equipped with the norm

$$|v|_{W^{1,p}} = \left( \int_{\Omega} |\nabla v|^p dx \right)^{1/p}.$$

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<sup>3)</sup>Recall  $\mathbf{E} = -\nabla\varphi$ ;  $\mathbf{J} = \sigma(u, |\mathbf{E}|\mathbf{E})$ .

Let  $X$  denote a real normed space with norm  $|\cdot|_X$  and let  $X^*$  be its dual space. By  $\langle x^*, x \rangle_X$  we denote the dual pairing between  $x^* \in X^*$  and  $x \in X$ . The symbol  $L^p(0, T, X)$  ( $1 \leq p \leq +\infty$ ) stands for the vector space of all strongly measurable mappings  $u : ]0, T[ \rightarrow X$  such that the function  $t \mapsto |u(t)|_X$  is in  $L^p(0, T)$  (cf. [4, Chap. III, §3; Chap. IV, §3], [5, App.], [11, Chap. 1]). For  $1 \leq p < +\infty$ , the spaces  $L^p(0, T; L^p(\Omega))$  and  $L^p(Q_T)$  are linearly isometric. Therefore, in what follows we identify these spaces.

Let  $H$  be a real Hilbert space with scalar product  $(\cdot, \cdot)_H$  such that  $X \subset H$  densely and continuously. Identifying  $H$  with its dual space  $H^*$  via Riesz' Representation Theorem, we obtain the continuous embedding  $H \subset X^*$  and

$$\langle h, x \rangle_X = (h, x)_H \quad \forall h \in H, \forall x \in X.$$

Given any  $u \in L^1(0, T; X)$  we identify this function with a function in  $L^1(0, T; X^*)$  and denote it again by  $u$ . If there exists  $U \in L^1(0, T; X^*)$  such that

$$\int_0^T u(t) \alpha'(t) dt \stackrel{\text{in } X^*}{=} - \int_0^T U(t) \alpha(t) dt \quad \forall \alpha \in C_c^\infty(]0, T[),$$

then  $U$  will be called derivative of  $u$  in the sense of distributions from  $]0, T[$  into  $X^*$  and denoted by  $u'$  (see [5, App.], [11, Chap. 21]).  $\square$

Let  $1 < p < +\infty$  be fixed. We make the following assumptions on the coefficients  $\sigma$ ,  $\kappa$  and the right hand side  $f$  in (1)–(2):

$$\begin{aligned} \text{(H1)} \quad & \begin{cases} \sigma : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \text{ is continuous,} \\ c_1 \tau^p - c_2 \leq \sigma(u, \tau) \tau^2, \quad 0 \leq \sigma(u, \tau) \leq c_3(1 + \tau^2)^{(p-2)/2} \\ \forall (u, \tau) \in \mathbb{R} \times \mathbb{R}_+, \text{ where } c_1, c_3 = \text{const} > 0 \text{ and } c_2 = \text{const} \geq 0; \end{cases} \\ \text{(H2)} \quad & \begin{cases} \kappa : \mathbb{R} \rightarrow \mathbb{R}_+ \text{ is continuous,} \\ 0 < \kappa_0 \leq \kappa(u) \leq \kappa_1 \quad u \in \mathbb{R}, \text{ where } \kappa_0, \kappa_1 = \text{const,} \end{cases} \end{aligned}$$

and the natural growth condition (with respect to (H1))

$$\text{(H3)} \quad \begin{cases} f : Q_T \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}_+ \text{ is Carathéodory,} \\ 0 \leq f(x, t, u, \xi) \leq c_4(1 + |\xi|^p) \\ \forall (x, t, u, \xi) \in Q_T \times \mathbb{R} \times \mathbb{R}^n, \text{ where } c_4 = \text{const} > 0. \end{cases}$$

It is readily seen that (H1) and (H3) are satisfied by the prototypes for  $\sigma$  and  $f$  we have considered in Section 1.

**Definition.** Assume (H1)–(H3) and suppose that the data in (3)–(5) satisfy

$$(10) \quad \varphi_D \in L^p(0, T; W^{1,p}(\Omega));$$

$$(11) \quad g = \text{const}, \quad h = \text{const};$$

$$(12) \quad u_0 \in L^1(\Omega).$$

The pair

$$(\varphi, u) \in L^p(0, T; W^{1,p}(\Omega)) \times L^q(0, T; W^{1,q}(\Omega)) \quad \left(1 < q < \frac{n+2}{n+1}\right)$$

is called weak solution of (1)–(5) if

$$(13) \quad \exists u' \in L^1(0, T; (W^{1,q'}(\Omega))^*);$$

$$(14) \quad \int_{Q_T} \sigma(u, |\nabla \varphi|) \nabla \varphi \cdot \nabla \zeta dx dt = 0 \quad \forall \zeta \in L^p(0, T; W_{\Gamma_D}^{1,p}(\Omega));$$

$$(15) \quad \left. \begin{aligned} & \int_0^T \langle u', v \rangle_{W^{1,q'}} dt + \int_{Q_T} \kappa(u) \nabla u \cdot \nabla v dx dt + g \int_0^T \int_{\partial \Omega} (u - h) v d_x S dt \\ & = \int_{Q_T} f(x, t, u, \nabla \varphi) v dx dt \quad \forall v \in L^\infty(0, T; W^{1,q'}(\Omega)); \end{aligned} \right\}$$

$$(16) \quad \varphi = \varphi_D \quad \text{a.e. on } \Sigma_D;$$

$$(17) \quad u(0) = u_0 \quad \text{in } (W^{1,q'}(\Omega))^*.$$

The condition  $1 < q < \frac{n+2}{n+1}$  is standard for weak solutions of parabolic equations with right hand side in  $L^1$ .

We notice that the function  $f(\cdot, \cdot, u, \nabla \varphi)v$  under the integral sign on the right hand side in (15) is in  $L^1(Q_T)$ . Indeed, (H3) gives  $f(\cdot, \cdot, u, \nabla \varphi) \in L^1(Q_T)$  while  $v \in L^\infty(0, T; W^{1,q'}(\Omega))$  can be identified with a function in  $L^\infty(Q_T)$ . To see this, we take any  $r \geq q'$  and obtain

$$\int_{\Omega} |v(x, t)|^r dx \leq \gamma_0^r \|v(t)\|_{W^{1,q'}}^r \text{mes } \Omega \quad \text{for a.e. } t \in [0, T],$$

where  $\gamma_0$  denotes the embedding constant of  $W^{1,q'}(\Omega) \subset C(\overline{\Omega})$  (notice  $q' > n + 2$ ). Thus,

$$\|v\|_{L^\infty(Q_T)} = \lim_{r \rightarrow \infty} \left( \int_{Q_T} |v(x, t)|^r dx dt \right)^{1/r} \leq \gamma_0 \|v\|_{L^\infty(0, T; W^{1,q'})} < +\infty.$$

To make precise the meaning of (17), let  $\frac{2n}{n+2} < q < \frac{n+2}{n+1}$  (cf. our main result below). We obtain the dense and continuous embeddings  $W^{1,q}(\Omega) \subset L^{nq/(n-q)}(\Omega) \subset L^2(\Omega)$ . Identifying  $L^2(\Omega)$  with its dual space, it follows  $L^2(\Omega) \subset (W^{1,q'}(\Omega))^*$  continuously (since  $W^{1,q'}(\Omega) \subset W^{1,q}(\Omega)$  continuously). Thus,

$$u \in L^q(0, T; (W^{1,q'}(\Omega))^*), \quad u' \in L^1(0, T; (W^{1,q'}(\Omega))^*) \quad (\text{cf. (13)}).$$

Hence, there exists  $\tilde{u} \in C([0, T]; (W^{1,q'}(\Omega))^*)$  such that  $\tilde{u}(t) = u(t)$  for a.e.  $t \in [0, T]$ . Then (17) with initial datum (12) is meant in the sense

$$\langle \tilde{u}(0), z \rangle_{W^{1,q'}} = \int_{\Omega} u_0(x) z(x) dx \quad \forall z \in W^{1,q'}(\Omega).$$

**Remark 2.** Let  $(\varphi, u)$  be a sufficiently regular classical solution of (1)–(5). More specifically, let  $u \in C^1(\overline{Q_T})$ . Then the function  $t \mapsto u(\cdot, t)$  possesses a distributional derivative  $u' \in L^2(0, T; L^2(\Omega))$  and there holds

$$\int_0^T \langle u', v \rangle_{L^2} dt = \int_{Q_T} \frac{\partial u}{\partial t} v dx dt \quad \forall v \in L^2(0, T; L^2(\Omega)).$$

By routine arguments one obtains that  $(\varphi, u)$  satisfies (14) and (15). Thus, (13)–(17) represents a weak formulation of (1)–(5).  $\square$

The main result of our paper is the following

**Theorem.** Assume (H1)–(H3). In addition to (H1), suppose that

$$(18) \quad (\sigma(u, \tau)\tau - \sigma(u, \bar{\tau})\bar{\tau})(\tau - \bar{\tau}) > 0 \quad \forall u \in \mathbb{R}, \quad \forall \tau, \bar{\tau} \in \mathbb{R}_+, \quad \tau \neq \bar{\tau}.$$

Further, let (10) and (12) be satisfied, and let

$$(19) \quad g = \text{const} > 0, \quad h = \text{const}.$$

(cf. (11)). Then there exists a pair

$$(\varphi, u) \in L^p(0, T; W^{1,p}(\Omega)) \times \left( \bigcap_{1 < q < (n+2)/(n+1)} L^q(0, T; W^{1,q}(\Omega)) \right)$$

such that (14) and (16) hold, and (13), (15) and (17) hold for every  $1 < q < \frac{n+2}{n+1}$ . Moreover,  $u$  satisfies

$$(20) \quad \|u\|_{L^\infty(0,T;L^1)} + \lambda \int_{Q_T} \frac{|\nabla u|^2}{(1+|u|)^{1+\lambda}} dxdt \leq c(1 + \|u_0\|_{L^1} + \|\varphi_D\|_{L^p(0,T;W^{1,p})}^p);$$

$$(21) \quad u \in \bigcap_{1 < r < (n+2)/n} L^r(0, T; L^r(\Omega)).$$

**Remark 3.** (Cf. (18)). For  $a : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  the following two statements about strict monotonicity are equivalent

- (i)  $(a(\tau)\tau - a(\bar{\tau})\bar{\tau})(\tau - \bar{\tau}) > 0 \quad \forall \tau, \bar{\tau} \in \mathbb{R}_+, \quad \tau \neq \bar{\tau};$
- (ii)  $(a(|\xi|)\xi - a(|\bar{\xi}|)\bar{\xi}) \cdot (\xi - \bar{\xi}) > 0 \quad \forall \xi, \bar{\xi} \in \mathbb{R}^n, \quad \xi \neq \bar{\xi} \quad (n \geq 2).$

This can be easily verified by elementary calculations.

We notice that the strict monotonicity of the function  $\xi \mapsto \sigma_0(u)(\delta + |\xi|^2)^{(p-2)/2}\xi$  ( $\delta > 0$ ,  $1 < p < +\infty$ ; cf. (6)) [as well as of the function  $\xi \mapsto \sigma_0(u)|\xi|^{p-2}\xi$  ( $2 \leq p < +\infty$ )] follows from the inequalities

$$\begin{aligned} & ((\delta + |\xi|^2)^{(p-2)/2}\xi - (\delta + |\bar{\xi}|^2)^{(p-2)/2}\bar{\xi}) \cdot (\xi - \bar{\xi}) \\ & \geq \begin{cases} \frac{p-1}{(\delta_0 + |\xi|^2 + |\bar{\xi}|^2)^{(2-p)/2}} |\xi - \bar{\xi}|^2 & \forall 0 \leq \delta \leq \delta_0, \quad \forall 1 < p \leq 2, \\ \min \left\{ \frac{1}{2}, \frac{1}{2^{p-2}} \right\} |\xi - \bar{\xi}|^p & \forall \delta \geq 0, \quad \forall 2 \leq p < +\infty \end{cases} \end{aligned}$$

(cf. [19, (I) p.71, p. 74]). The coefficient  $\min \left\{ \frac{1}{2}, \frac{1}{2^{p-2}} \right\}$  is related to the inequality

$$\alpha^{p-2} + \beta^{p-2} \geq \min \left\{ 1, \frac{1}{2^{p-3}} \right\} (\alpha + \beta)^{p-2} \quad \forall \alpha, \beta \in \mathbb{R}_+, \quad \forall 2 \leq p < +\infty$$

(note by P.-A. Ivert).

**Remark 4.** For  $\sigma(u, \tau) = \sigma_0(u)$  (i.e.,  $p = 2$  in (6)),  $f(x, t, u, \xi) = \sigma_0(u)|\xi|^2$  (cf. (9)) and Dirichlet boundary conditions, in [1] ( $n = 3$ ) and [9] ( $n = 2$ ) the authors proved the existence of a weak solution  $(\varphi, u)$  of (1)–(5) such that

$$(\varphi, u) \in L^2(0, T; W^{1,2}(\Omega)) \times L^2(0, T; W^{1,2}(\Omega)).$$

### 3. PROOF OF THE MAIN RESULT

We divide the proof into three parts.

**3.1. Existence of an approximate solution.** For  $\varepsilon > 0$  we define the Carathéodory function

$$f_\varepsilon(x, t, u, \xi) = \frac{f(x, t, u, \xi)}{1 + \varepsilon f(x, t, u, \xi)}, \quad (x, t, u, \xi) \in Q_T \times \mathbb{R} \times \mathbb{R}^n.$$

Let  $(u_{0,\varepsilon})_{\varepsilon>0}$  be a sequence of functions in  $L^2(\Omega)$  such that  $u_{0,\varepsilon} \rightarrow u_0$  strongly in  $L^1(\Omega)$  as  $\varepsilon \rightarrow 0$ . We have

**Lemma 1.** *For every  $\varepsilon > 0$  there exists a pair*

$$(\varphi_\varepsilon, u_\varepsilon) \in L^p(0, T; W^{1,p}(\Omega)) \times L^2(0, T; W^{1,2}(\Omega))$$

*such that*

$$(22) \quad \int_{Q_T} \sigma(u_\varepsilon, |\nabla \varphi_\varepsilon|) \nabla \varphi_\varepsilon \cdot \nabla \zeta \, dx dt = 0 \quad \forall \zeta \in L^p(0, T; W_{\Gamma_D}^{1,p}(\Omega));$$

$$(23) \quad \varphi_\varepsilon = \varphi_D \quad \text{a.e. on } \Sigma_D;$$

$$(24) \quad \exists u'_\varepsilon \in L^2(0, T; (W^{1,2}(\Omega))^*);$$

$$(25) \quad \left. \begin{aligned} & \int_0^T \langle u'_\varepsilon, v \rangle_{W^{1,2}} dt + \int_{Q_T} \kappa(u_\varepsilon) \nabla u_\varepsilon \cdot \nabla v \, dx dt + g \int_0^T \int_{\partial\Omega} (u_\varepsilon - h) v \, d_x S dt \\ & = \int_{Q_T} f_\varepsilon(x, t, u_\varepsilon, \nabla \varphi_\varepsilon) v \, dx dt \quad \forall v \in L^2(0, T; W^{1,2}(\Omega)); \end{aligned} \right\}$$

$$(26) \quad u_\varepsilon(\cdot, 0) = u_{0,\varepsilon} \quad \text{a.e. in } \Omega.$$

By routine arguments it is readily seen that (25) is equivalent to

$$(27) \quad \left. \begin{aligned} & \langle u'_\varepsilon(t), z \rangle_{W^{1,2}} + \int_{\Omega} \kappa(u_\varepsilon(x, t)) \nabla u_\varepsilon(x, t) \cdot \nabla z(x) \, dx \\ & + g \int_{\partial\Omega} (u_\varepsilon(x, t) - h) z(x) \, d_x S \\ & = \int_{\Omega} f_\varepsilon(x, t, u_\varepsilon(x, t), \nabla \varphi_\varepsilon(x, t)) z(x) \, dx \end{aligned} \right\}$$

for a.e.  $t \in [0, T]$  and all  $z \in W^{1,2}(\Omega)$ , where the set of measure zero of those  $t$  for which (27) fails, does not depend on  $z$ .

*Proof of Lemma 1.* We prove this lemma by the aid of Schauder's Fixed Point Theorem.

*Step 1. Construction of a mapping*

$$\mathcal{T} : \overline{B}_R \rightarrow \overline{B}_R,$$

where

$$\overline{\mathcal{B}}_R = \{w \in L^2(0, T; L^2(\Omega)); \|w\|_{L^2(L^2)} \leq R\}^{4)}$$

( $R > 0$  suitable choosen). For this we need the following two preliminary results 1° and 2°.

1° *Given any  $u \in L^2(0, T; L^2(\Omega))$ , there exists exactly one  $\varphi \in L^p(0, T; W^{1,p}(\Omega))$  ( $\varphi = \varphi_u$ ) such that*

$$(28) \quad \int_{Q_T} \sigma(u, |\nabla \varphi|) \nabla \varphi \cdot \nabla \zeta \, dxdt = 0 \quad \forall \zeta \in L^p(0, T; W_{\Gamma_D}^{1,p}(\Omega));$$

$$(29) \quad \varphi = \varphi_D \quad \text{a.e. on } \Sigma_D.$$

To prove this, we define a mapping

$$\mathcal{A} : L^p(0, T; W_{\Gamma_D}^{1,p}(\Omega)) \longrightarrow L^{p'}(0, T; (W_{\Gamma_D}^{1,p}(\Omega))^*) \quad (\mathcal{A} = \mathcal{A}_u)$$

by

$$\langle \mathcal{A}(\psi), \zeta \rangle_{L^p(W_{\Gamma_D}^{1,p})} = \int_{Q_T} \sigma(u, |\nabla(\psi + \varphi_D)|) \nabla(\psi + \varphi_D) \cdot \nabla \zeta \, dxdt,$$

where  $\psi, \zeta \in L^p(0, T; W_{\Gamma_D}^{1,p}(\Omega))$ . From (H1) and (10) it follows that this mapping maps bounded sets into bounded sets. By (18) [cf. also Remark 4] we have

$$\langle \mathcal{A}(\psi) - \mathcal{A}(\overline{\psi}), \psi - \overline{\psi} \rangle_{L^p(W_{\Gamma_D}^{1,p})} > 0$$

for all  $\psi, \overline{\psi} \in L^p(0, T; W_{\Gamma_D}^{1,p})$ ,  $\psi \neq \overline{\psi}$ . Finally, appealing once more to (H1) we obtain the coercivity of  $\mathcal{A}$ .

The theory of monotone operators yields the existence and uniqueness of an  $\omega \in L^p(0, T; W_{\Gamma_D}^{1,p}(\Omega))$  such that

$$\mathcal{A}(\omega) = 0$$

(see, e.g., [20, Chap. 2.2], [29, Chap. 26.2]). Then the function  $\varphi = \omega + \varphi_D$  is in  $L^p(0, T; W^{1,p}(\Omega))$  and solves (28)–(29).

2° *Let  $u \in L^2(0, T; L^2(\Omega))$  and let  $\varphi = \varphi_u \in L^2(0, T; W^{1,2}(\Omega))$  denote the uniquely determined solution of (28)–(29) (cf. 1°). Then there exists exactly one*

$$\hat{u} \in L^2(0, T; W^{1,2}(\Omega)) \cap C([0, T]; L^2(\Omega)) \quad (\hat{u} = \hat{u}_{\varphi_u})$$

such that

$$(30) \quad \exists \hat{u}' \in L^2(0, T; (W^{1,2}(\Omega))^*);$$

$$(31) \quad \left. \begin{aligned} & \int_0^T \langle \hat{u}', v \rangle_{W^{1,2}} dt + \int_{Q_T} \kappa(u) \nabla \hat{u} \cdot \nabla v \, dxdt + g \int_0^T \int_{\partial\Omega} (\hat{u} - h) v \, d_x S dt \\ & = \int_{Q_T} f_\varepsilon(x, t, u, \nabla \varphi) v \, dxdt \quad \forall v \in L^2(0, T; W^{1,2}(\Omega)); \end{aligned} \right\}$$

$$(32) \quad \hat{u}(\cdot, 0) = u_{0,\varepsilon} \quad \text{a.e. in } \Omega;$$

---

<sup>4)</sup>In what follows, for indexes we write  $L^p(L^p)$  in place of  $L^p(0, T; L^p(\Omega))$ ,  $L^p(W_D^{1,p})$  etc.



$$(33) \quad \|\hat{u}\|_{L^\infty(L^2)} + \|\hat{u}\|_{L^2(W^{1,2})} + \|\hat{u}'\|_{L^2((W^{1,2})^*)} \leq c,$$

where the constant  $c$  depends on  $\kappa_0, \kappa_1, g, h$  (see (H2), (19)),  $\|u_{0,\varepsilon}\|_{L^2}$  and  $\frac{1}{\varepsilon}$ , but is independent of  $u$ .

This result follows from the theory of linear evolution equations (see, e.g., [12, Chap. 7.1]). To see this, it suffices to notice that

$$[w, z]_{W^{1,2}} = \int_{\Omega} \nabla w \cdot \nabla z \, dx + \int_{\partial\Omega} wz \, d_x S, \quad w, z \in W^{1,2}(\Omega)$$

is a scalar product on  $W^{1,2}(\Omega)$  which is equivalent to the standard scalar product on this space.  $\square$

From (33) we conclude that there exists a constant  $R > 0$  which depends the same quantities as the constant  $c$  such that  $\|\hat{u}\|_{L^2(L^2)} \leq R$ . We now define a mapping

$$\mathcal{T} : \overline{\mathcal{B}}_R \longrightarrow \overline{\mathcal{B}}_R$$

by

$$\mathcal{T}u = \hat{u}, \quad \hat{u} \text{ according to } 2^\circ.$$

$\square$

*Step 2. Properties of  $\mathcal{T}$*  We have

3°  $\mathcal{T}(\overline{\mathcal{B}}_R)$  is precompact;

4°  $\mathcal{T}$  is continuous.

*Proof of 3°* Let  $(w_k) \subset \mathcal{T}(\overline{\mathcal{B}}_R)$  ( $k \in \mathbb{N}$ ) be any sequence. Then  $w_k = \mathcal{T}u_k = \hat{u}_k$ , where  $u_k \in \overline{\mathcal{B}}_R$ . By (33),

$$\|w_k\|_{L^2(W^{1,2})} + \|w_k'\|_{L^2((W^{1,2})^*)} \leq c \quad \forall k \in \mathbb{N}.$$

The embedding  $W^{1,2}(\Omega) \subset L^2(\Omega)$  being dense and compact, a well-known compactness theorem (see [20, pp. 58–59]) yields the existence of a subsequence of  $(w_k)$  (not relabelled) such that

$$w_k \rightarrow w \quad \text{strongly in } L^2(0, T; L^2(\Omega)) \quad \text{as } k \rightarrow \infty.$$

*Proof of 4°* Let  $(u_k) \subset \overline{\mathcal{B}}_R$  ( $k \in \mathbb{N}$ ) be a sequence such that  $u_k \rightarrow u$  strongly in  $L^2(0, T; L^2(\Omega))$  as  $k \rightarrow \infty$ . By passing to a subsequence if necessary, we may assume

$$(34) \quad u_k \rightarrow u \quad \text{a.e. in } Q_T \quad \text{as } k \rightarrow \infty.$$

Let  $\varphi_k, \varphi \in L^p(0, T; W^{1,p}(\Omega))$  ( $\varphi_k = \varphi_{u_k}, \varphi = \varphi_u; k \in \mathbb{N}$ ) be determined by 1°, i.e.,

$$(35) \quad \int_{Q_T} \sigma(u_k, |\nabla \varphi_k|) \nabla \varphi_k \cdot \nabla \zeta \, dx dt = 0 \quad \forall \zeta \in L^p(0, T; W_{\Gamma_D}^{1,p}(\Omega)),$$

$$(36) \quad \varphi_k = \varphi_D \quad \text{a.e. on } \Sigma_D,$$

and  $\varphi$  satisfies (28)–(29).

Next, let  $\hat{u}_k, \hat{u} \in L^2(0, T; W^{1,2}(\Omega))$  ( $k \in \mathbb{N}$ ) be determined by 2°, i.e.,  $\hat{u}_k$  satisfies (30)–(33) in place of  $\hat{u}$ . We show

$$(37) \quad \mathcal{T}u_k = \hat{u}_k \longrightarrow \hat{u} = \mathcal{T}u \quad \text{strongly in } L^2(0, T; L^2(\Omega)) \quad \text{as } k \rightarrow \infty.$$

To this end, let us assume

$$(38) \quad \nabla \varphi_k \longrightarrow \nabla \varphi \quad \text{a.e. in } Q_T \text{ as } k \rightarrow \infty$$

(the proof will be given below). We insert  $v = \hat{u}_k - \hat{u}$  into the variational identities in (31) for  $\hat{u}_k$  and  $\hat{u}$ , respectively, and form the difference of both identities. This gives an integral relation which contains the term

$$\int_0^T \langle \hat{u}'_k - \hat{u}', \hat{u}_k - \hat{u} \rangle_{W^{1,2}} dt = \frac{1}{2} \|\hat{u}_k(T) - \hat{u}(T)\|_{L^2}^2$$

(observe (32)) and the right hand side

$$\int_{Q_T} (f_\varepsilon(x, t, u_k, \nabla \varphi_k) - f_\varepsilon(x, t, u, \nabla \varphi)) (\hat{u}_k - \hat{u}) dx dt.$$

By (34) and (38),

$$\lim_{k \rightarrow \infty} \int_{Q_T} (f_\varepsilon(x, t, u_k, \nabla \varphi_k) - f_\varepsilon(x, t, u, \nabla \varphi))^2 dx dt = 0.$$

The claim (37) is proved.

*Proof of (38)* By (36), the function  $\zeta = \varphi - \varphi_D$  is admissible in (35). Combining (H1) and Hölder's inequality we obtain  $\|\varphi_k\|_{L^p(W^{1,p})} \leq \text{const}$  for all  $k \in \mathbb{N}$ . Hence, there exists a subsequence of  $(\varphi_k)$  (not relabelled) such that

$$\varphi_k \longrightarrow \chi \quad \text{weakly in } L^p(0, T; W^{1,p}(\Omega)) \quad \text{as } k \rightarrow \infty.$$

It follows  $\chi = \varphi_D$  a.e. on  $\Sigma_D$ . Observing (34), the passage to the limit  $k \rightarrow \infty$  in (35) is easily carried out by the monotonicity trick (see, e.g., [20, p. 172], [29, p. 474]) to obtain

$$\int_{Q_T} \sigma(u, |\nabla \chi|) \nabla \chi \cdot \nabla \zeta dx dt = 0 \quad \forall \zeta \in L^p(0, T; W_{\Gamma_D}^{1,p}(\Omega)).$$

Thus,  $\chi$  satisfies (28)–(29) in place of  $\varphi$ . By the strict monotonicity of  $\xi \mapsto \sigma(u, |\xi|)\xi$  (cf. (18) resp. Remark 3) we obtain  $\chi = \varphi$ , and the whole sequence  $(\varphi_k)$  converges weakly in  $L^p(0, T; W^{1,p}(\Omega))$  to  $\varphi$ . Therefore,

$$\lim_{k \rightarrow \infty} \int_{Q_T} [\sigma(u_k, |\nabla \varphi_k|) \nabla \varphi_k - \sigma(u_k, |\nabla \varphi|) \nabla \varphi] \cdot \nabla (\varphi_k - \varphi) dx dt = 0.$$

Finally, for a.e.  $(x, t) \in Q_T$ , define

$$\begin{aligned} E_k(x, t) &= [\sigma(u_k(x, t), |\nabla \varphi_k(x, t)|) \nabla \varphi_k(x, t) \\ &\quad - \sigma(u_k(x, t), |\nabla \varphi(x, t)|) \nabla \varphi(x, t)] \cdot \nabla (\varphi_k(x, t) - \varphi(x, t)). \end{aligned}$$

We obtain

$$E_k(x, t) \longrightarrow 0 \quad \text{for a.e. } (x, t) \in Q_T \text{ as } k \rightarrow \infty.$$

A well-known argument due to Leray-Lions [17] now gives (38) (cf. also, [20, pp. 184–185], [2], [3], [23], [24]).

*Step 3. Existence of a fixed point of  $\mathcal{T}$*  The Schauder Fixed Point Theorem yields the existence of an element  $u_\varepsilon \in \overline{\mathcal{B}}_R$  such that  $\mathcal{T}u_\varepsilon = u_\varepsilon$ . We then determine  $\varphi_\varepsilon = \varphi_{u_\varepsilon}$  according to 1° above. The pair  $(\varphi_\varepsilon, u_\varepsilon)$  satisfies (22)–(26).

**3.2. A-priori estimates.** We have

**Lemma 2.** *Let be  $(\varphi_\varepsilon, u_\varepsilon)$  as in Lemma 1.*

$$(39) \quad \|\varphi_\varepsilon\|_{L^p(W^{1,p})} \leq c(1 + \|\varphi_D\|_{L^p(W^{1,p})})^{5);$$

$$(40) \quad \int_{Q_T} f_\varepsilon(x, t, u_\varepsilon, \nabla \varphi_\varepsilon) dx dt \leq c(1 + \|\varphi_D\|_{L^p(W^{1,p})}^p);$$

$$(41) \quad \left. \begin{aligned} & \|u_\varepsilon\|_{L^\infty(L^1)} + \lambda \int_{Q_T} \frac{|\nabla u_\varepsilon|^2}{(1 + |u_\varepsilon|)^{1+\lambda}} dx dt \\ & \leq c(1 + \|u_{0,\varepsilon}\|_{L^1} + \|\varphi_D\|_{L^p(W^{1,p})}^p), \quad 0 < \lambda < 1; \end{aligned} \right\}$$

$$(42) \quad \|u_\varepsilon\|_{L^q(W^{1,q})} \leq c(q) \quad \forall 1 < q < \frac{n+2}{n+1};$$

$$(43) \quad \|u_\varepsilon\|_{L^r(L^r)} \leq c(r) \quad \forall 1 < r < \frac{n+2}{n};$$

$$(44) \quad \|u'_\varepsilon\|_{L^1((W^{1,q'})^*)} \leq c(q) \quad \forall 1 < q < \frac{n+2}{n+1},$$

where  $c(q) \rightarrow +\infty$  as  $q \rightarrow \frac{n+2}{n+1}$ , and  $c(r) \rightarrow +\infty$  as  $r \rightarrow \frac{n+2}{n}$ .

*Proof of Lemma 2.* By (23) the function  $\zeta = \varphi_\varepsilon - \varphi_D$  is admissible in (22). The estimate (39) is then easily obtained by (H1) and Hölder's inequality. From (H1) and (H3) it follows that

$$(45) \quad f(x, t, u, \xi) \leq c_5(1 + \sigma(u, |\xi|)|\xi|^2) \quad (c_5 = \text{const} > 0)$$

for all  $(x, t, u, \xi) \in Q_T \times \mathbb{R} \times \mathbb{R}^n$ . With the help of this inequality the estimate in (40) is easily deduced from (39).

To prove (41), for  $s \in \mathbb{R}$  and  $0 < \lambda < 1$  we define the functions

$$\begin{aligned} \Phi(s) &= \Phi_\lambda(s) = \left(1 - \frac{1}{(1 + |s|)^\lambda}\right) \text{sign } s^{6), \\ \Psi(s) &= \Psi_\lambda(s) = |s| + \frac{1}{1 - \lambda}(1 - (1 + |s|)^{1-\lambda}). \end{aligned}$$

We obtain

$$\Phi'(s) = \frac{\lambda}{(1 + |s|)^{1+\lambda}}, \quad \Psi'(s) = \Phi(s), \quad \frac{|s|}{2} - \frac{2^{(1-\lambda)/2}}{1 - \lambda} \leq \Psi(s) \leq |s|,$$

and

$$\nabla \Phi(u_\varepsilon) = \Phi'(u_\varepsilon) \nabla u_\varepsilon = \lambda \frac{\nabla u_\varepsilon}{(1 + |u_\varepsilon|)^{1+\lambda}} \quad \text{for a.e. } (x, t) \in Q_T,$$

$$\int_0^t \langle u'_\varepsilon, \Phi(u_\varepsilon) \rangle_{W^{1,2}} ds = \int_\Omega \Psi(u_\varepsilon(x, t)) dx - \int_\Omega \Psi(u_{0,\varepsilon}(x)) dt \quad \forall t \in [0, T]$$

---

<sup>5)</sup>In what follows, by  $c$  we denote constants which may change their numerical value from line to line but do not depend on  $\varepsilon$ .

<sup>6)</sup> $\text{sign}(0) = 0$ .

(cf. [22], [23], [24]). We insert  $z = \Phi(u_\varepsilon(\cdot, t))$  into (27), integrate over the interval  $[0, t]$  and make use of (40). By elementary calculations we obtain (41).

The proof of (42) is now easily done by well-known arguments (see, e.g., [2], [3], [23], [24]). Indeed, let  $1 < q < n$ . A simple application of Hölder's inequality yields

$$(46) \quad \|z\|_{L^{q(n+1)/n}} \leq \|z\|_{L^1}^{1/(n+1)} \|z\|_{L^{nq/(n-q)}}^{n/(n+1)} \quad \forall z \in L^{nq/(n-q)}(\Omega).$$

Next, given  $1 < q < \frac{n+2}{n+1}$  we set  $\lambda = \frac{1}{n}(n+2-q(n+1))$ . Using the integral estimate in (41) one finds

$$\begin{aligned} \int_{Q_T} |\nabla u_\varepsilon|^q dx dt &= \int_{Q_T} \frac{|\nabla u_\varepsilon|^q}{(1+|u_\varepsilon|)^{q(1+\lambda)/2}} \cdot (1+|u_\varepsilon|)^{q(1+\lambda)/2} dx dt \\ &\leq \frac{c}{\lambda^{q/2}} \left( \int_{Q_T} (1+|u_\varepsilon|)^{q(n+1)/n} dx dt \right)^{(2-q)/2}. \end{aligned}$$

To estimate the integral on the right hand side, we take  $z = u_\varepsilon(\cdot, t)$  in (46) and use then the bound on  $\|u_\varepsilon\|_{L^\infty(L^1)}$  in (41) and the Sobolev embedding theorem  $W^{1,q}(\Omega) \subset L^{nq/(n-q)}(\Omega)$ . We obtain

$$\int_{Q_T} (1+|u_\varepsilon|)^{q(n+1)/n} dx dt \leq c \left( 1 + \int_{Q_T} |\nabla u_\varepsilon|^q dx dt \right).$$

Whence (42). Using once more (46), we get (43).

We finally prove (44). From (27) it follows for a.e.  $t \in [0, T]$  and all  $z \in W^{1,q'}(\Omega)$  that

$$|\langle u'_\varepsilon(t), z \rangle_{W^{1,q'}}| \leq c(1 + \|u_\varepsilon(t)\|_{W^{1,q}} + \|\varphi_\varepsilon(t)\|_{W^{1,p}}^p) \|z\|_{W^{1,q'}},$$

where the constant  $c$  does not depend on  $\varepsilon$ . By (39) and (40), the function in parantheses is uniformly bounded independently of  $\varepsilon$ . The estimate (44) is now easily seen.

**3.3. Passage to the limit  $\varepsilon \rightarrow 0$ .** We begin by proving the existence of convergent subsequences of  $(\varphi_\varepsilon, u_\varepsilon)$ . Then we complete the proof of our main result by showing that the limit functions of these subsequences yield a weak solution of (1)–(5).

**Lemma 3.** *Let be  $(\varphi_\varepsilon, u_\varepsilon)$  as in Lemma 1. Then there exists a subsequence (not relabelled) such that*

$$(47) \quad \varphi_\varepsilon \rightharpoonup \varphi \quad \text{weakly in } L^p(0, T; W^{1,p}(\Omega));$$

$$(48) \quad \left. \begin{aligned} u_\varepsilon &\rightharpoonup u \quad \text{weakly in } L^q(0, T; W^{1,q}(\Omega)) \quad \left(1 < q < \frac{n+2}{n+1}\right) \\ \text{and weakly in } L^r(0, T; L^r(\Omega)) &\quad \left(1 < r < \frac{n+2}{n}\right); \end{aligned} \right\}$$

$$(49) \quad u_\varepsilon \rightarrow u \quad \text{a.e. in } Q_T;$$

$$(50) \quad \nabla \varphi_\varepsilon \rightarrow \nabla \varphi \quad \text{a.e. in } Q_T;$$

$$(51) \quad \int_{Q_T} |\sigma(u_\varepsilon, |\nabla \varphi_\varepsilon|)|\nabla \varphi_\varepsilon|^2 - \sigma(u, |\nabla \varphi|)|\nabla \varphi|^2| dx dt \rightarrow 0$$

as  $\varepsilon \rightarrow 0$ .

*Proof of Lemma 3.* The existence of subsequences of  $(\varphi_\varepsilon, u_\varepsilon)$  satisfying (47), (48) follows from the reflexivity of the respective spaces.

We prove (49). To this end, take  $q$  such that  $\frac{2}{n+2} < q < \frac{n+2}{n+1}$ . Then  $W^{1,q}(\Omega) \subset L^2(\Omega)$  compactly (recall  $n = 2$  or  $n = 3$ ). We identify  $L^2(\Omega)$  with its dual space and obtain the continuous embedding  $L^2(\Omega) \subset (W^{1,q'}(\Omega))^*$  (cf. Section 2). Observing the bounds on  $u_\varepsilon$  and  $u'_\varepsilon$  in (42) and (44), respectively, from the compactness result in [6, Prop. 1] resp. [26, Cor. 4] we obtain  $u_\varepsilon \rightarrow u$  strongly in  $L^q(0, T; L^2(\Omega))$ , and thus

$$u_\varepsilon \rightarrow u \quad \text{a.e. in } Q_T \quad \text{as } \varepsilon \rightarrow 0$$

(again by passing to a subsequence if necessary). With the help of this convergence of  $(u_\varepsilon)$  we find (50) by the same arguments as for the proof of (38).

It remains to prove (51). From (H1) and (39) it follows that the sequence  $(\sigma(u_\varepsilon, |\nabla \varphi_\varepsilon|) \nabla \varphi_\varepsilon)$  is bounded in  $[L^{p'}(Q_T)]^n$  for all  $\varepsilon > 0$ . We therefore may assume that

$$\sigma(u_\varepsilon, |\nabla \varphi_\varepsilon|) \nabla \varphi_\varepsilon \rightharpoonup F \quad \text{weakly in } [L^{p'}(\Omega)]^n \quad \text{as } \varepsilon \rightarrow 0.$$

Then (22) implies

$$\lim_{\varepsilon \rightarrow 0} \int_{Q_T} \sigma(u_\varepsilon, |\nabla \varphi_\varepsilon|) |\nabla \varphi_\varepsilon|^2 dx dt = \int_{Q_T} F \cdot \nabla \varphi dx dt.$$

On the other hand, for all  $G \in [L^p(\Omega)]^n$  and a.e.  $(x, t) \in Q_T$ ,

$$(\sigma(u_\varepsilon, |G|)G - \sigma(u_\varepsilon, |\nabla \varphi_\varepsilon|) \nabla \varphi_\varepsilon) \cdot (G - \nabla \varphi_\varepsilon) \geq 0.$$

Integrating this inequality over  $Q_T$ , letting  $\varepsilon \rightarrow 0$  and using the monotonicity trick (within the context of the dual pairing  $(L^p(Q_T), L^{p'}(Q_T))$ ) we get

$$\int_{Q_T} \sigma(u, |\nabla \varphi|) |\nabla \varphi|^2 dx dt = \int_{Q_T} F \cdot \nabla \varphi dx dt.$$

Thus,

$$(52) \quad \lim_{\varepsilon \rightarrow 0} \int_{Q_T} \sigma(u_\varepsilon, |\nabla \varphi_\varepsilon|) |\nabla \varphi_\varepsilon|^2 dx dt = \int_{Q_T} \sigma(u, |\nabla \varphi|) |\nabla \varphi|^2 dx dt.$$

In addition, by (49) and (50),

$$(53) \quad \sigma(u_\varepsilon, |\nabla \varphi_\varepsilon|) |\nabla \varphi_\varepsilon|^2 \rightarrow \sigma(u, |\nabla \varphi|) |\nabla \varphi|^2 \quad \text{a.e. in } Q_T$$

as  $\varepsilon \rightarrow 0$ . Then (51) follows from (52) and (53) by the aid of Lebesgue's Dominated Convergence Theorem.  $\square$

*Completion of the proof of the main result.* We prove that the pair

$$(\varphi, u) \in L^p(0, T; W^{1,p}(\Omega)) \times \left( \bigcap_{1 < q < (n+2)/(n+1)} L^q(0, T; W^{1,q}(\Omega)) \right)$$

obtained by Lemma 3, fulfills all conditions stated in our main theorem.

The passage to the limit  $\varepsilon \rightarrow 0$  in (22), (23) gives (14), (16), respectively. The estimates in (20) as well as the integrability property (21) are easily derived from (41) and (43), respectively, and using (49).

We prove that  $(\varphi, u)$  satisfies (13), (15) and (17). We take  $z \in W^{1,q'}(\Omega)$  ( $1 < q < \frac{n+2}{n+1}$ ) in (27), multiply each term by  $\alpha \in C^1([0, T])$ ,  $\alpha(T) = 0$ , and integrate over the interval  $[0, T]$ . It follows

$$\begin{aligned}
 & - \int_{Q_T} u_\varepsilon z \alpha' dx dt + \int_{Q_T} \kappa(u_\varepsilon) \nabla u_\varepsilon \cdot \nabla z \alpha dx dt + g \int_0^T \int_{\partial\Omega} (u_\varepsilon - h) z \alpha d_x S dt \\
 (54) \quad & = \int_{\Omega} u_{0,\varepsilon}(x) z(x) dx \alpha(0) + \int_{Q_T} f_\varepsilon(x, t, u_\varepsilon, \nabla \varphi_\varepsilon) z \alpha dx dt.
 \end{aligned}$$

The passage to the limit  $\varepsilon \rightarrow 0$  for the second integral on the right hand side of (54) is easily done as follows. More generally, for all  $w \in L^\infty(Q_T)$  we have

$$(55) \quad \lim_{\varepsilon \rightarrow 0} \int_{Q_T} f_\varepsilon(x, t, u_\varepsilon, \nabla \varphi_\varepsilon) w dx dt = \int_{Q_T} f(x, t, u, \nabla \varphi) w dx dt.$$

Indeed, we use once more (45) to obtain

$$(56) \quad f(x, t, u_\varepsilon, \nabla \varphi_\varepsilon) |w| \leq c_5 \|w\|_{L^\infty} (1 + \sigma(u_\varepsilon, |\nabla \varphi_\varepsilon|) |\nabla \varphi_\varepsilon|^2)$$

for a.e.  $(x, t) \in Q_T$  and all  $\varepsilon > 0$ . Integrating this inequality over  $Q_T$  and using (39) we find

$$\varepsilon f(x, t, u_\varepsilon, \nabla \varphi_\varepsilon) \longrightarrow 0 \quad \text{for a.e. } (x, t) \in Q_T \quad \text{as } \varepsilon \rightarrow 0.$$

It follows

$$\lim_{\varepsilon \rightarrow 0} f_\varepsilon(x, t, u_\varepsilon, \nabla \varphi_\varepsilon) w = f(x, t, u, \nabla \varphi) w \quad \text{for a.e. } (x, t) \in Q_T.$$

In addition, since  $f_\varepsilon \leq f$ , (56) implies that the function on the right hand side is an integrable bound for  $f_\varepsilon |w|$  a.e. in  $Q_T$ . Moreover, by (51), these functions converge in  $L^1(Q_T)$  when  $\varepsilon \rightarrow 0$ . The claim (55) thus follows from Lebesgue's Dominated Convergence Theorem.

The passage to the limit  $\varepsilon \rightarrow 0$  in (54) now gives

$$\begin{aligned}
 & - \int_{Q_T} u z \alpha' dx dt + \int_{Q_T} \kappa(u) \nabla u \cdot \nabla z \alpha dx dt + g \int_0^T \int_{\partial\Omega} (u - h) z \alpha d_x S dt \\
 (57) \quad & = \int_{\Omega} u_0(x) z(x) dx \alpha(0) + \int_{Q_T} f(x, t, u, \nabla \varphi) z \alpha dx dt
 \end{aligned}$$

(recall  $u_{0,\varepsilon} \rightarrow u_0$  strongly in  $L^1(\Omega)$  as  $\varepsilon \rightarrow 0$ ). We prove the existence of a distributional derivative of  $u$ . For this, we define a mapping  $F : [0, T] \rightarrow (W^{1,q'}(\Omega))^*$  by

$$\begin{aligned}
 \langle F(t), z \rangle_{W^{1,q'}} &= - \int_{\Omega} \kappa(u(x, t)) \nabla u(x, t) \cdot \nabla z(x) dx - g \int_{\partial\Omega} (u(x, t) - h) z(x) d_x S \\
 &+ \int_{\Omega} f(x, t, u(x, t), \nabla \varphi(x, t)) z(x) dx, \quad z \in W^{1,q'}(\Omega).
 \end{aligned}$$

It follows

$$\|F(t)\|_{(W^{1,q'})^*} \leq c (1 + \|u(t)\|_{W^{1,q}} + \|\varphi(t)\|_{W^{1,p}}^p)$$

for a.e.  $t \in [0, T]$  (cf. the proof of (44)). The function  $t \mapsto \langle F(t), z \rangle_{W^{1,q'}}$  being measurable for all  $z \in W^{1,q'}(\Omega)$ ,  $F$  is strongly measurable on  $[0, T]$  by virtue of Pettis' Theorem. Thus,

$$F \in L^1(0, T; (W^{1,q'}(\Omega))^*).$$

We take  $\alpha \in C_c^\infty(]0, T[)$  in (57) and rewrite this variational identity in the form

$$\left\langle -\int_0^T u(t)\alpha'(t)dt, z \right\rangle_{W^{1,q'}} = \left\langle \int_0^T F(t)\alpha(t)dt, z \right\rangle_{W^{1,q'}}.$$

This implies the existence of the distributional derivative  $u' \in L^1(0, T; (W^{1,q'}(\Omega))^*)$  and

$$\int_0^T u'(t)\alpha(t)dt \stackrel{\text{in } (W^{1,q'})^*}{=} \int_0^T F(t)\alpha(t)dt \quad \forall \alpha \in C_c^\infty(]0, T[).$$

It follows that there exists  $\tilde{u} \in C([0, T]; (W^{1,q'}(\Omega))^*)$  such that  $\tilde{u}(t) = u(t)$  for a.e.  $t \in [0, T]$  and

$$\left\langle \int_0^T u'(t)\alpha(t)dt, z \right\rangle_{W^{1,q'}} = -\langle \tilde{u}(0)\alpha(0), z \rangle_{W^{1,q'}} - \int_{Q_T} u z \alpha' dx dt$$

for all  $\alpha \in C^1([0, T])$ ,  $\alpha(T) = 0$ . Now (15) and (17) are easily obtained by standard arguments.

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